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# Algebraic Approach to solve $t\bar{t}$ dilepton equations

**Lars Sonnenschein**
*LPNHE Universités Paris VI et VII*
*France*
*E-mail:* sonne@in2p3.fr

The set of non-linear equations describing the Standard Model kinematics of the top quark antiqark production system in the dilepton decay channel has at most a four-fold ambiguity due to two not fully reconstructed neutrinos. Its most precise and robust solution is of major importance for measurements of top quark properties like the top quark mass and $t\bar{t}$ spin correlations. Simple algebraic operations allow to transform the non-linear equations into a system of two polynomial equations with two unknowns. These two polynomials of multidegree eight can in turn be analytically reduced to one polynomial with one unknown by means of resultants. The obtained univariate polynomial is of degree sixteen and the coefficients are free of any singularity. The number of its real solutions is determined analytically by means of Sturm's theorem, which is as well used to isolate each real solution into a unique pairwise disjoint interval. The solutions are polished by seeking the sign change of the polynomial in a given interval through binary bracketing. Further a new Ansatz - exploiting an accidental cancelation in the process of transforming the equations - is presented. It permits to transform the initial system of equations into two polynomial equations with two unknowns. These two polynomials of multidegree two can be reduced to one univariate polynomial of degree four by means of resultants. The obtained quartic equation can be solved analytically. The analytical solution has singularities which can be circumvented by the algebraic approach described above.

*International Workshop on Top Quark Physics*
*January 12-15, 2006*
*Coimbra, Portugal*

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PoS(TOP2006)037

## 1. Introduction

In 1992, Dalitz and Goldstein published a numerical method based on geometrical considerations to solve the system of equations describing the kinematics of the $t\bar{t}$ decay in the dilepton channel [0] [0]. In 2004 an approximation of the system of equations - assuming that the transverse momentum of the $t\bar{t}$ system can be neglected - has been solved analytically [0] by means of computer algebra software such as Maple [0]. Meanwhile the transverse momentum constraint has been omitted while the solution is still derived by means of computer algebra and its accuracy does not reach real precision [0]. Recently, the system of equations could be solved algebraically to real precision free of any singularity [0] based on Sturm's theorem. This algebraic approach is discussed in detail here. Further the analytical solution based on a new Ansatz which minimizes the amount of intermediate steps to derive the solution is presented [0]. This approach makes the need of computer algebra superfluous. In addition it provides more transparency and control over singularities which are intrinsic to the analytical solution. Irreducible singularities can be circumvented in exploiting the analytical Ansatz of the algebraic approach [0]. The accuracy achieved is - as already in the algebraic approach - of real precision. Important improvements in terms of robustness, code volume and time consumption with respect to the algebraic approach make this method more convenient for applications in practice. Other solution methods can compare their performance to the algebraic reference methods described here. It should be mentioned that different approachs leading to analytical solutions, without giving a complete algebraic derivation and without rigorous discussion of reducible and irreducible singularities exist in the literature [0] [0].

In the next section the system of $t\bar{t}$ dilepton equations is introduced, followed by a description of the algebraic solution exploiting Sturm's theorem and its implementation as algorithm. Subsequently the derivation of the analytical solution is explained including a rigorous discussion of the reducible and irreducible singularities. Afterwards the performance of the two methods which is consistent with each other is elaborated.

## 2. $t\bar{t}$ dilepton kinematics

The system of equations describing the kinematics of $t\bar{t}$ dilepton events can be expressed by the two linear and six non linear equations

$$\begin{aligned}
\not{E}_x &= p_{\nu_x} + p_{\bar{\nu}_x} \\
\not{E}_y &= p_{\nu_y} + p_{\bar{\nu}_y} \\
E_\nu^2 &= p_{\nu_x}^2 + p_{\nu_y}^2 + p_{\nu_z}^2 \\
E_{\bar{\nu}}^2 &= p_{\bar{\nu}_x}^2 + p_{\bar{\nu}_y}^2 + p_{\bar{\nu}_z}^2 \\
m_{W^+}^2 &= (E_{\ell^+} + E_\nu)^2 - (p_{\ell_x^+} + p_{\nu_x})^2 \\
&\quad -(p_{\ell_y^+} + p_{\nu_y})^2 - (p_{\ell_z^+} + p_{\nu_z})^2 \\
m_{W^-}^2 &= (E_{\ell^-} + E_{\bar{\nu}})^2 - (p_{\ell_x^-} + p_{\bar{\nu}_x})^2 \\
&\quad -(p_{\ell_y^-} + p_{\bar{\nu}_y})^2 - (p_{\ell_z^-} + p_{\bar{\nu}_z})^2 \\
m_t^2 &= (E_b + E_{\ell^+} + E_\nu)^2 - (p_{b_x} + p_{\ell_x^+} + p_{\nu_x})^2
\end{aligned} \tag{2.1}$$



PoS(TOP2006)037

$$
\begin{aligned}
& -(p_{b_y} + p_{\ell_y^+} + p_{\nu_y})^2 - (p_{b_z} + p_{\ell_z^+} + p_{\nu_z})^2 \\
m_t^2 \;=\; & (E_{\bar{b}} + E_{\ell^-} + E_{\bar{\nu}})^2 - (p_{\bar{b}_x} p_{\ell_x^-} + p_{\bar{\nu}_x})^2 \\
& -(p_{\bar{b}_y} + p_{\ell_y^-} + p_{\bar{\nu}_y})^2 - (p_{\bar{b}_z} + p_{\ell_z^-} + p_{\bar{\nu}_z})^2 \,.
\end{aligned}
$$

The $z$-axis is here assumed to be parallel orientated to the beam axis while the $x$- and $y$-coordinates span the transverse plane. The first two equations relate the projection of the missing transverse energy onto one of the transverse axes ($x$ or $y$) to the sum of the neutrino and antineutrino momentum components belonging to the same projection. The next two equations relate the energy of the neutrino and antineutrino, which are assumed to be massless in good approximation, with their momenta. Finally four non linear equations describe the $W$ boson and top quark (antiquark) mass constraints by relating the invariant masses to the energy and momenta of their decay particles via relativistic 4-vector arithmetics.

## 3. Algebraic solution

This system of equations can be reduced to four equations by simply substituting in the last four equations the neutrino and antineutrino energies by the third and fourth equations and substituting the antineutrino transverse momenta by the first two equations solved to these momenta. In this way the four unknowns $p_{\nu_x}$, $p_{\nu_y}$, $p_{\nu_z}$ and $p_{\bar{\nu}_z}$ are left. One pair of equations, describing the $t \to bW^+ \to b\ell^+\nu_\ell$ parton branch of the event, depends on $p_{\nu_z}$ while the other pair of equations, describing the $\bar{t} \to \bar{b}W^- \to \bar{b}\ell^-\bar{\nu}_\ell$ parton branch of the event, depends on $p_{\bar{\nu}_z}$. By means of ordinary algebraic operations both pairs can be solved to the longitudinal neutrino and antineutrino momentum $p_{\nu_z}$ and $p_{\bar{\nu}_z}$ respectively. The equations can be written in the form

$$
\begin{aligned}
p_{\nu_z} &= a_1 \pm \sqrt{a_1^2 + a_2} \\
p_{\nu_z} &= b_1 \pm \sqrt{b_1^2 + b_2}
\end{aligned}
\tag{3.1}
$$

for the top quark parton branch and

$$
\begin{aligned}
p_{\bar{\nu}_z} &= c_1 \pm \sqrt{c_1^2 + c_2} \\
p_{\bar{\nu}_z} &= d_1 \pm \sqrt{d_1^2 + d_2}
\end{aligned}
\tag{3.2}
$$

for the anti-top quark parton branch with the coefficients

$$
\begin{aligned}
a_1 \;&=\; a_{11} + a_{12} p_{\nu_x} + a_{13} p_{\nu_y} \\
a_2 \;&=\; a_{21} + a_{22} p_{\nu_x} + a_{23} p_{\nu_y} + a_{24} p_{\nu_x}^2 + a_{25} p_{\nu_x} p_{\nu_y} + a_{26} p_{\nu_y}^2
\end{aligned}
\tag{3.3}
$$

and $b$ equivalent for the first pair of equations (3.1). For the second pair of equations (3.2) holds analogically

$$
\begin{aligned}
c_1 \;&=\; c_{11} + c_{12} p_{\bar{\nu}_x} + c_{13} p_{\bar{\nu}_y} \\
c_2 \;&=\; c_{21} + c_{22} p_{\bar{\nu}_x} + c_{23} p_{\bar{\nu}_y} + c_{24} p_{\bar{\nu}_x}^2 + c_{25} p_{\bar{\nu}_x} p_{\bar{\nu}_y} + c_{26} p_{\bar{\nu}_y}^2
\end{aligned}
\tag{3.4}
$$



PoS(TOP2006)037

and $d$ equivalent. The explicit expressions in terms of the initial equations (2.1) are given in [0]. After equating both equations of each pair, there remain two equations with the two unknowns $p_{\nu_x}$ and $p_{\nu_y}$.

Again by means of ordinary algebraic operations the two non linear equations can be transformed into two polynomials of multi-degree eight. To solve these two polynomials to $p_{\nu_x}$ the resultant with respect to the neutrino momentum $p_{\nu_y}$ is computed as follows. The coefficients and monomials of the two polynomials are rewritten in such a way that they are ordered in powers of $p_{\nu_y}$ like

$$\begin{aligned} f &= f_1 p_{\nu_y}^4 + f_2 p_{\nu_y}^3 + f_3 p_{\nu_y}^2 + f_4 p_{\nu_y} + f_5 \\ g &= g_1 p_{\nu_y}^4 + g_2 p_{\nu_y}^3 + g_3 p_{\nu_y}^2 + g_4 p_{\nu_y} + g_5 \end{aligned} \tag{3.5}$$

where $f$ and $g$ are polynomials of the remaining unknowns $p_{\nu_x}$, $p_{\nu_y}$ and the coefficients $f_m$, $g_n$ are univariate polynomials of $p_{\nu_x}$. The resultant can then be obtained by computing the determinant of the Sylvester matrix

$$\mathrm{Res}(p_{\nu_y}) = \mathrm{Det}\begin{pmatrix} f_1 & & & g_1 & & \\ f_2 & f_1 & & g_2 & g_1 & \\ f_3 & f_2 & f_1 & g_3 & g_2 & g_1 \\ f_4 & f_3 & f_2 & f_1 & g_4 & g_3 & g_2 & g_1 \\ f_5 & f_4 & f_3 & f_2 & g_5 & g_4 & g_3 & g_2 \\ & f_5 & f_4 & f_3 & & g_5 & g_4 & g_3 \\ & & f_5 & f_4 & & & g_5 & g_4 \\ & & & f_5 & & & & g_5 \end{pmatrix} = 0 \tag{3.6}$$

which is equated to zero. The omitted elements of the matrix are identical to zero. Since each element in the matrix is a polynomial itself the evaluation is very elaborative. There are two ways to compute the determinant in practice. The more elegant way from a programming technical point of view is to invoke recursively a function which computes subdeterminants and consists of a very limited number of lines. Unfortunately it turns out that this approach is too time consuming. The other way is to let Maple [0] compute and optimize the determinant as a function of the unknown $p_{\nu_x}$ and implement it. This way the code grows orders of magnitude in size but on the other hand the evaluation speeds up by orders of magnitude.

The resultant is a univariate polynomial of the form

$$\begin{aligned} 0 &= h_1 p_{\nu_x}^{16} + h_2 p_{\nu_x}^{15} + h_3 p_{\nu_x}^{14} + h_4 p_{\nu_x}^{13} + h_5 p_{\nu_x}^{12} + h_6 p_{\nu_x}^{11} \\ &+ h_7 p_{\nu_x}^{10} + h_8 p_{\nu_x}^{9} + h_9 p_{\nu_x}^{8} + h_{10} p_{\nu_x}^{7} + h_{11} p_{\nu_x}^{6} \\ &+ h_{12} p_{\nu_x}^{5} + h_{13} p_{\nu_x}^{4} + h_{14} p_{\nu_x}^{3} + h_{15} p_{\nu_x}^{2} + h_{16} p_{\nu_x} + h_{17} \end{aligned} \tag{3.7}$$

with the remaining unknown $p_{\nu_x}$. Fig. 1 shows such a polynomial for a given event. It is of degree 16 and analytical solutions of general univariate polynomials are only known until degree four. Abel's impossibility theorem and Galois demonstrated that a univariate polynomial



PoS(TOP2006)037

of degree five can in general not be solved analytically with a finite number of additions, subtractions, multiplications, divisions, and root extractions [0]. Thus from here on the solutions of the univariate polynomial (3.7) have to be obtained by different means. In principle the problem can be reduced to an Eigenvalue problem. Unfortunately, in practice it turns out that the implementation of the Eigenvalue package in Root [0] gives only reasonable solutions for univariate polynomials of degree 14 and below. Finally the number of solutions is obtained analytically by applying Sturm's theorem [0] which consists of building a sequence of univariate polynomials $h(p_{\nu_x}), h'(p_{\nu_x}), h_a(p_{\nu_x}), h_b(p_{\nu_x}), ..., h_m(p_{\nu_x}) = \text{const.}$, where $h'$ is the first derivative of the univariate polynomial $h$ with respect to $p_{\nu_x}$ and the following polynomials are the remainders of a long division of their immediate left neighbour polynomial divided by the next left neighbour polynomial. The sequence ends when the last polynomial is a constant. In the case the constant vanishes, the initial polynomial has at least one multiple real root which can be splitted by long division through the last non constant polynomial in the Sturm sequence. In this case one solution is already known. The sequence is evaluated at two neutrino momenta $p_{\nu_{x_{1,2}}}$ (initially at the kinematic limits) and the difference between the number of sign changes of the evaluated sequence at the two interval limits is determined. The obtained quantity corresponds to the number of real solutions in the given interval.

This means that the theorem of Jacques Charles François Sturm - which he has proven in 1829 [0] - is extremely powerful since in the case of no real solutions no time needs to be spent for the unsuccessful attempt to find one.

To reduce numerical inaccuracies, all polynomial evaluations are applied using Horner's rule which factors out powers of the polynomial variable $p_{\nu_x}$ [0]. Further the solutions are separated by applying Sturm's theorem with varying interval boundaries. Once the solutions are separated in unique pairwise disjoint intervals they are polished by binary bracketing exploiting the knowledge about the sign change at the root in the given interval. This is possible since it is guaranteed that there is only one single solution in a given interval per construction (Now one could turn the way to solve a given Eigenvalue problem the other way around and use the Sturm sequence to solve the characteristic polynomial to obtain the Eigenvalues). Once the solutions are found - most frequently there are two but never more than four - they can be inserted in equations (3.5). Such that these equations reduce to two univariate polynomials of degree four which in turn can be solved

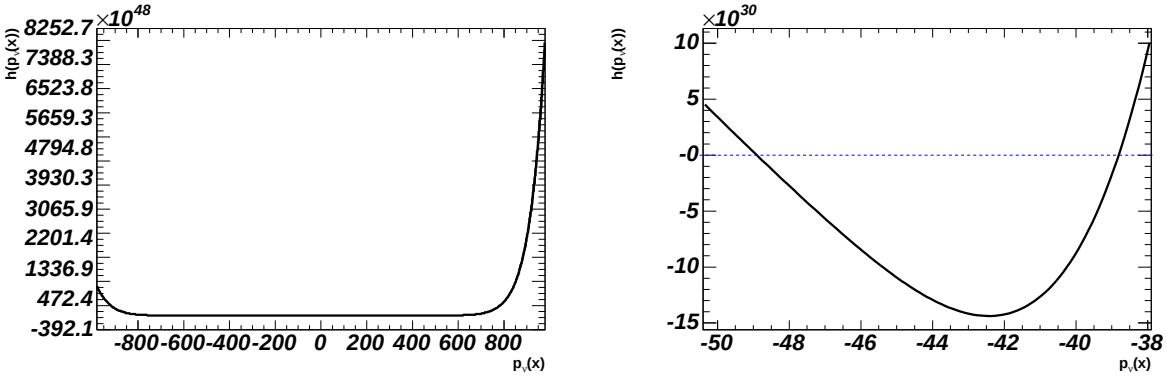


**Figure 1:** A typical univariate polynomial of degree 16 whose real roots in $p_{\nu_x}$ are solutions of the initial system of equations describing the $t\bar{t}$ dilepton kinematics. The right plot shows the polynomial zoomed around the interesting $p_{\nu_x}$ range of the abscissa where two solutions are located.



PoS(TOP2006)037

analytically to $p_{\nu_y}$ with a four fold ambiguity. The ambiguities can be eliminated in requiring the roots of these two polynomials to coincide since both equations have to be satisfied simultaneously. $p_{\bar{\nu}_x}$ and $p_{\bar{\nu}_y}$ can be simply determined with help of the first two equations in (2.1). To determine the longitudinal neutrino and antineutrino momenta $p_{\nu_z}$ and $p_{\bar{\nu}_z}$ the equations (3.1) and (3.2) can be evaluated respectively. Again the two-fold ambiguity, here due to the square root sign, can be resolved in requiring the solutions to coincide simultaneously for both equations of one parton branch.

## 4. Analytical solution

The system of equations (2.1) can again be subdivided in two entangled sets of equations. One set of equations, describing the $t \to bW^+ \to b\ell^+\nu_\ell$ parton branch of the event, depends on $p_{\nu_z}$ while the other pair of equations, describing the $\bar{t} \to \bar{b}W^- \to \bar{b}\ell^-\bar{\nu}_\ell$ parton branch of the event, depends on $p_{\bar{\nu}_z}$.

The equation describing the invariance of the $W$ boson mass can be expressed in the following way

$$\begin{aligned} m_{W^+}^2 &= (E_{\ell^+} + E_\nu)^2 - (\vec{p_{\ell^+}} + \vec{p}_\nu)^2 \\ &= E_{\ell^+}^2 + 2E_{\ell^+}E_\nu + E_\nu^2 - \vec{p_{\ell^+}}^2 - 2\vec{p_{\ell^+}}\vec{p}_\nu - \vec{p}_\nu{}^2 \\ &= m_{\ell^+}^2 + 2E_{\ell^+}E_\nu - 2\vec{p_{\ell^+}}\vec{p}_\nu \end{aligned} \tag{4.1}$$

which can be rewritten as

$$E_\nu = \frac{m_{W^+}^2 - m_{\ell^+}^2 + 2\vec{p_{\ell^+}}\vec{p}_\nu}{2E_{\ell^+}}. \tag{4.2}$$

The equation describing the invariance of the top quark mass can be transformed in the same way leading to

$$E_\nu = \frac{m_t^2 - m_b^2 - m_{\ell^+}^2 - 2E_bE_{\ell^+} + 2\vec{p}_b\vec{p_{\ell^+}} + 2(\vec{p}_b + \vec{p_{\ell^+}})\vec{p}_\nu}{2(E_b + E_{\ell^+})} \tag{4.3}$$

where additional terms emerge due to the fact that quantities which depended in equation (4.2) only on the lepton depend now also on the $b$ quark. Next the unknown $E_\nu$ can be eliminated by subtracting equation (4.3) from (4.2), leading to an equation of the form

$$0 = a_1 + a_2 p_{\nu_x} + a_3 p_{\nu_y} + a_4 p_{\nu_z} \tag{4.4}$$

where the coefficients $a$ are constants given in [0]. This equation is linear in the three neutrino momentum components. Since the unknown $p_{\nu_z}$ does only appear in the top quark parton branch it is mandatory to eliminate this variable with a linear independent equation of the top quark parton branch to obtain finally together with the equations of the antitop quark branch two equations of the two unknowns $p_{\nu_x}$ and $p_{\nu_y}$.

To eliminate the unknown $p_{\nu_z}$ it is straight forward to use equation (4.2) (for convenience multiplied by the denominator $2E_{\ell^+}$). The neutrino energy $E_\nu$ can be expressed in terms of the



PoS(TOP2006)037

three neutrino momenta components in substituting it with the third equation of (2.1). To obtain a polynomial equation the squared of this equation is being considered in the following. The terms squared in the longitudinal neutrino momentum cancel out accidentally. It is exactly this cancellation which permits to eliminate the neutrino momentum $p_{\nu_z}$ by means of the linear equation (4.4). The resulting equation of the form

$$0 = c_{22} + c_{21} p_{\nu_x} + c_{11} p_{\nu_y} + c_{21} p_{\nu_x}^2 + c_{10} p_{\nu_x} p_{\nu_y} + c_{00} p_{\nu_y}^2 \tag{4.5}$$

is a multivariate polynomial of multidegree two which depends only on the transverse neutrino momenta $p_{\nu_x}$ and $p_{\nu_y}$. The coefficients are again constants which can be expressed in terms of the former derived constants $a$ and are given in [0].

In the same way can be proceeded for the equations describing the antitop quark parton branch. The equivalent of equation (4.4) reads

$$0 \;=\; b_1 + b_2 p_{\bar{\nu}_x} + b_3 p_{\bar{\nu}_y} + b_4 p_{\bar{\nu}_z} \tag{4.6}$$

and the counter part of polynomial (4.5) can be written as

$$0 = d'_{22} + d'_{21} p_{\bar{\nu}_x} + d'_{11} p_{\bar{\nu}_y} + d'_{21} p_{\bar{\nu}_x}^2 + d'_{10} p_{\bar{\nu}_x} p'_{\bar{\nu}_y} + d'_{00} p_{\bar{\nu}_y}^2 \,. \tag{4.7}$$

The two equations linear in the three (anti-)neutrino momenta (4.4) and (4.6) build the minimal Ansatz used here. In contrast the non-minimal Ansatz made in [0], [0] is based on two equations linear in the four unknowns $p_{\bar{\nu}_x}$, $p_{\bar{\nu}_y}$, $p_{\bar{\nu}_z}$, $p_{\nu_z}$.

To reduce equations (4.5) and (4.7) to two polynomial equations of two unknowns the transverse antineutrino momenta of equation (4.7) can be expressed by the transverse neutrino momenta with help of the missing transverse energy relations of the system of equations (2.1). Since these relations are linear in the neutrino and antineutrino momenta the substitution leads again to a polynomial of the form

$$0 = d_{22} + d_{21} p_{\nu_x} + d_{11} p_{\nu_y} + d_{21} p_{\nu_x}^2 + d_{10} p_{\nu_x} p_{\nu_y} + d_{00} p_{\nu_y}^2 \tag{4.8}$$

with multidegree two whose coefficients are given in [0]. To solve these two polynomials without loss of generality to $p_{\nu_x}$ the resultant with respect to the neutrino momentum $p_{\nu_y}$ is computed as follows. The coefficients and monomials of the two polynomials (4.5) and (4.8) are rewritten in such a way that they are ordered in powers of $p_{\nu_y}$ like

$$c \;=\; c_0 p_{\nu_y}^2 + c_1 p_{\nu_y} + c_2, \tag{4.9}$$

$$d \;=\; d_0 p_{\nu_y}^2 + d_1 p_{\nu_y} + d_2 \tag{4.10}$$

where $c$ and $d$ are polynomials of the remaining unknowns $p_{\nu_x}$, $p_{\nu_y}$ and the coefficients $c_m$, $d_n$ are univariate polynomials of $p_{\nu_x}$ only. The coefficients contain two reducible singularities which can be removed by multiplying them with the least common multiple of the factors which can become singular. The resultant can then be obtained by computing the determinant of the Sylvester matrix

$$\mathrm{Res}(p_{\nu_y}) = \mathrm{Det} \begin{pmatrix} c_0 & & d_0 & \\ c_1 & c_0 & d_1 & d_0 \\ c_2 & c_1 & d_2 & d_1 \\ & c_2 & & d_2 \end{pmatrix} = 0 \tag{4.11}$$



PoS(TOP2006)037

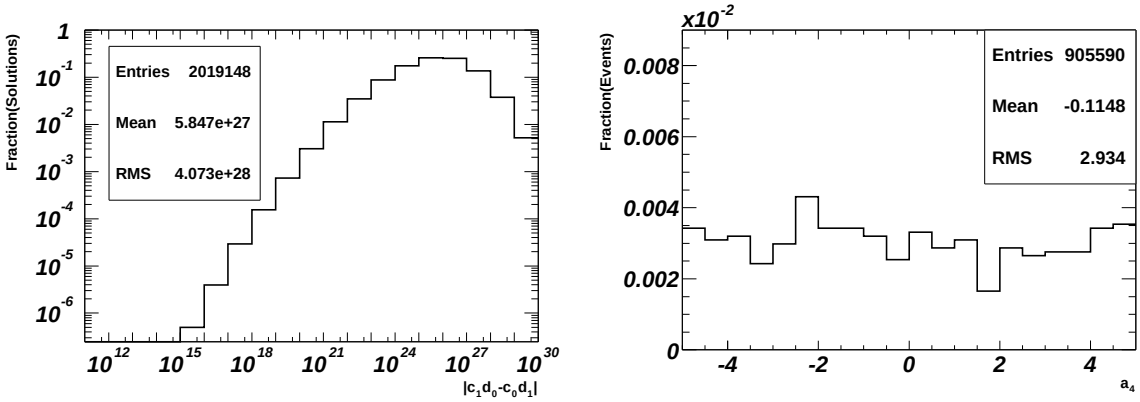


**Figure 2:** Left: Distribution of the expression $c_1d_0 - c_0d_1$ which appears in the denominator in the solution of $p_{\nu_y}$. Since the distribution is symmetric around zero the module of the expression is plotted. As can be seen the values assumed are far away from zero which would cause a singularity in the solution. To the right, the coefficient $a_4$ appearing in the equations describing the top quark parton branch is plotted. It is flat distributed over the whole phase space including the value zero where an irreducible singularity of its reciprocal resides. The coefficient $b_4$ of the antitop quark parton branch behaves in the same way.

which is equated to zero. The omitted elements of the matrix are identical to zero. The resultant is a univariate polynomial of the form

$$h_0 p_{\nu_x}^4 + h_1 p_{\nu_x}^3 + h_2 p_{\nu_x}^2 + h_3 p_{\nu_x} + h_4 \tag{4.12}$$

which contains the remaining unknown $p_{\nu_x}$. It is of degree four and can be solved analytically. The coefficients $h$ are given in [0]. This result shows that there is at most a four fold ambiguity. Here it has been assumed that the 4-vectors of the particles and the top quark and $W$ boson masses which enter into the system of equations are known exactly. Under these conditions there are two solutions in about 80% of cases and four solutions else. In the next section it will be investigated how this distribution changes under more realistic conditions when the assumption of exactness between particles and reconstructed objects is not valid anymore. Once the solution of a neutrino momentum $p_{\nu_x}$ has been found the other neutrino and antineutrino momentum components have to be determined. The antineutrino momentum $p_{\bar{\nu}_x}$ can be immediately obtained by the linear transverse missing energy relation of the initial system of equations (2.1). To derive the neutrino momentum $p_{\nu_y}$ equation (4.9) is multiplied by $d_0$ and equation (4.10) is multiplied by $c_0$ so that their difference yields a linear equation in the neutrino momentum $p_{\nu_y}$ which can then be isolated as

$$p_{\nu_y} = \frac{c_0 d_2 - c_2 d_0}{c_1 d_0 - c_0 d_1}. \tag{4.13}$$

Again the antineutrino momentum $p_{\bar{\nu}_y}$ can be immediately obtained by the corresponding linear transverse missing energy relation of the initial system of equations. As shown in Fig. 2, left plot, the coefficient in the denominator of equation (4.13) does not acquire values which are even close to the singularity at zero. Thus it is ensured that the neutrino momenta $p_{\nu_y}$ and $p_{\bar{\nu}_y}$ can be computed accurately over the whole phase space of possible solutions.



PoS(TOP2006)037

Finally the longitudinal (anti-)neutrino momenta $p_{\nu_z}$ and $p_{\bar{\nu}_z}$ can be easily obtained by the linear equations (4.4) and (4.6) assuming that the coefficients $a_4$ and $b_4$ are different from zero since they appear as a product together with the longitudinal (anti-)neutrino momenta themselves. The distributions of the coefficients are shown in Fig. 2, right plot. The fraction of solutions close to the singularity - irreducible in the analytical solution - is below the per mill level and may be neglected for practical purposes. From a theoretical point of view this singularity can be circumvented in solving the neutrino momenta $p_{\nu_z}$ and $p_{\bar{\nu}_z}$ analytically with the equations 3.1 and 3.2 of the algebraic approach which does not contain any singularity. It has been verified that the longitudinal (anti-)neutrino momentum does not typically vanish together with the coefficient $a_4$ ($b_4$) simultaneously, which would cause the singularity to disappear.

## 5. Performance

The performance studies discussed here are assuming Tevatron proton antiproton collider settings with a centre of mass energy of 1.96 TeV which has been set up in the Monte Carlo event generator PYTHIA 6.220 [0]. Cross-checks at a centre of mass energy of 14 TeV assuming the LHC proton collider environment confirm the independence of the method of particular collider settings.

The performance of the algebraic approach making use of Sturm's theorem can be confirmed by the analytical solution. In both cases one has to zoom very strong into the area of interest, close to a real solution, to be able to recognize a root of the univariate polynomial. The algebraic approach finds solutions in 99.9% of events assuming the particle momenta to be known exactly. The inefficiency, which is for practical purposes already a set of measure zero, reduces to the sub per mill level in the analytical solution due to less elaborative computational algebra involved. The neutrino momenta $p_\nu^{sol}$ of the solutions are compared to the generated ones $p_\nu^{gen}$ by defining a metric $\chi$ through

$$\begin{aligned}\chi^2 = &(p_{\nu_x}^{gen} - p_{\nu_x}^{sol})^2 + (p_{\nu_y}^{gen} - p_{\nu_y}^{sol})^2 + (p_{\nu_z}^{gen} - p_{\nu_z}^{sol})^2 \\ &+ (p_{\bar{\nu}_x}^{gen} - p_{\bar{\nu}_x}^{sol})^2 + (p_{\bar{\nu}_y}^{gen} - p_{\bar{\nu}_y}^{sol})^2 + (p_{\bar{\nu}_z}^{gen} - p_{\bar{\nu}_z}^{sol})^2 \,.\end{aligned} \tag{5.1}$$

The solutions coincide typically within real precision to the generated neutrino momenta. Fig. 3 left plot shows impressively how accurate and reliably the method is working. If the $W$ boson mass is generated off-shell while its pole mass is assumed in the solution, the efficiency drops to 89%. Relaxing the same assumption for the top quark mass results into a further decrease of efficiency to 84%. Beyond, an infrared-safe cone algorithm [0] with cone size $R = 0.5$ in the space spanned by pseudorapidity and azimuthal angle has been applied to the hadronic final state particles. Two reconstructed jets, two leptons and missing transverse energy are required for an event to be selected. The jets are accepted as $b$-tagged if they coincide within $\Delta R < 0.5$ with the generated $b$ quarks. The solution efficiency drops to 71% and can be re-established at 81% in solving both $b$ quark jet permutations. Smearing the leptons and jets with the energy resolution of the DØ detector [0] decreases the efficiency to 75%. In practice, a given event can be solved repeatedly, with the energy of the particles and objects smeared randomly within the detector resolution once each iteration to improve the solution efficiency. As already mentioned above these observations are consistent between the algebraic approach [0] and the analytical solution [0]. This confirms on



PoS(TOP2006)037

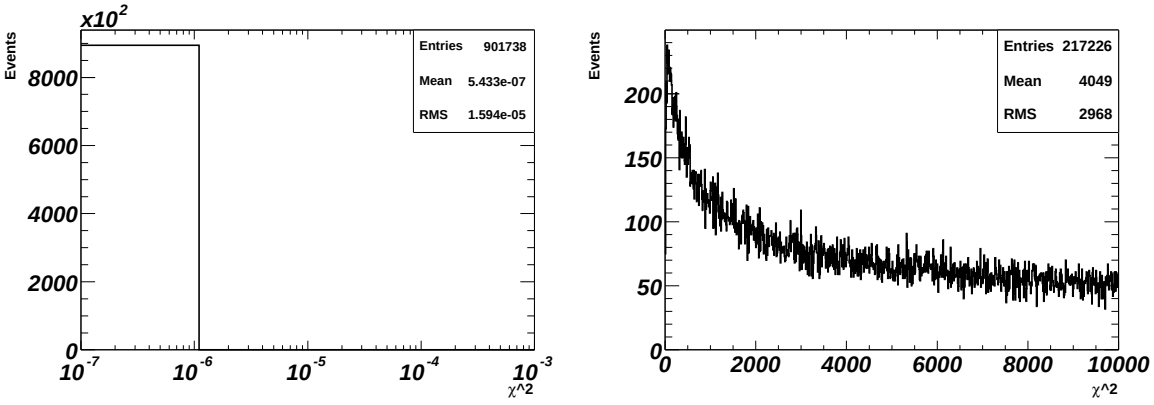


**Figure 3:** Solution $\chi^2$ defined as the difference between solved and generated neutrino momenta, added in quadrature, for the closest solution of each event. The left plot shows agreement to real precision obtained in the case of parton momenta inserted into the solution are known exactly. In the right plot reconstructed objects smeared in energy with DØ detector resolution yield degraded agreement.

one hand the reliability of the algebraic approach and rises on the other hand the question what numerical methods with a superior solution efficiency are actually solving.

Considering only events which could be solved it is important to investigate the number of solutions in dependence of the experimental settings since this number is directly proportional to the ambiguities of the solved and reconstructed events which in turn determines the significance of the solutions and any observable making use of it [0]. The fraction of solved events having exactly two solutions, the average number of solutions and its RMS is investigated for different experimental settings. First the particle final state is considered. Relaxing the amount of assumptions about the top quark and $W$ boson masses (pole mass inserted into solution while off-shell generated) increases the fraction of solved events with exactly two solutions while the average number of solutions and its RMS decrease slightly. Allowing both $b$ quark jet permutations - assuming that the charge of the quarks can not be determined with adequate certainty - the fraction of events having exactly two solutions drops considerably in favour of a higher solution multiplicity with larger RMS. Further the number of solutions for reconstructed objects has been investigated in evaluating the following sequence of conditions: First the right $b$ quark permutation and then both permutations have been solved. Then energy resolution smearing has been applied to the reconstructed objects and finally 100 solution iterations have been accomplished to take into account the uncertainty in the measured energy of the reconstructed objects. The general tendency is that the fraction of solved events with exactly two solutions decreases with less accurate knowledge about the particles and objects while the solution multiplicity and its RMS does increase.

## 6. Conclusions

An algebraic approach exploiting Sturm's theorem and the analytical solution of the system of equations describing the $t\bar{t}$ dilepton kinematics have been presented. Both solutions rely on the technique of resultants to reduce two multivariate polynomials to one univariate polynomial. This concept makes the use of symbolic computer algebra software to derive the solution superfluous.



PoS(TOP2006)037

The analytical solution has two reducible singularities which can be removed and two irreducible singularities which can be circumvented by exploiting the analytical Ansatz of the algebraic approach. The performance of the two methods is consistent with each other. The fraction of events whithout any solution or with no solution matching the generated (anti-)neutrino momenta with real precision - assuming that the particle momenta and masses inserted into the solution are known exactly - is a set of measure zero and can in very good approximation be neglected for practical purposes. Little deviations of the inserted particle momenta and masses from their true values drop the solution efficiency and purity considerably. At the same time the solution multiplicity increases. This raises the question what more efficient numerical methods are actually solving. General solution methods can compare their performance to the algebraic reference methods described here.

## Acknowledgments

Many thanks to Vladislav Simak, Petr Homola and Paul Russo for their support and useful discussions. I'm also grateful for helpful suggestions from colleagues of LPNHE, Universités de Paris VI, VII and the DØ collaboration. This work has been supported by the *Commissariat à l'Energie Atomique* and CNRS/*Institut National de Physique Nucléaire et de Physique des Particules*, France.

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PoS(TOP2006)037

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PoS(TOP2006)037