Algebraic Approach to solve $t\bar{t}$ Dilepton Equations

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- $t\bar{t}$ dilepton kinematics
- Algebraic Technique of Resultants
- Sturm's Theorem
- Analytical shortcut
- Conclusions

$t\bar{t}$ dilepton kinematics

$$t\bar{t} \to W^+ b W^- \bar{b} \to \ell^+ \nu b \ell^- \bar{\nu} \bar{b}$$

$$\begin{aligned}
\not{E}_x &= p_{\nu_x} + p_{\bar{\nu}_x}, \\
\not{E}_y &= p_{\nu_y} + p_{\bar{\nu}_y}, \\
E_\nu^2 &= p_{\nu_x}^2 + p_{\nu_y}^2 + p_{\nu_z}^2, \\
E_{\bar{\nu}}^2 &= p_{\bar{\nu}_x}^2 + p_{\bar{\nu}_y}^2 + p_{\bar{\nu}_z}^2, \\
m_{W^+}^2 &= (E_{\ell^+} + E_\nu)^2 - (p_{\ell_x^+} + p_{\nu_x})^2, \\
&\quad -(p_{\ell_y^+} + p_{\nu_y})^2 - (p_{\ell_z^+} + p_{\nu_z})^2, \\
m_{W^-}^2 &= (E_{\ell^-} + E_{\bar{\nu}})^2 - (p_{\ell_x^-} + p_{\bar{\nu}_x})^2, \\
&\quad -(p_{\ell_y^-} + p_{\bar{\nu}_y})^2 - (p_{\ell_z^-} + p_{\bar{\nu}_z})^2, \\
m_t^2 &= (E_b + E_{\ell^+} + E_\nu)^2 - (p_{b_x} + p_{\ell_x^+} + p_{\nu_x})^2, \\
&\quad -(p_{b_y} + p_{\ell_y^+} + p_{\nu_y})^2 - (p_{b_z} + p_{\ell_z^+} + p_{\nu_z})^2, \\
m_{\bar{t}}^2 &= (E_{\bar{b}} + E_{\ell^-} + E_{\bar{\nu}})^2 - (p_{\bar{b}_x} p_{\ell_x^-} + p_{\bar{\nu}_x})^2, \\
&\quad -(p_{\bar{b}_y} + p_{\ell_y^-} + p_{\bar{\nu}_y})^2 - (p_{\bar{b}_z} + p_{\ell_z^-} + p_{\bar{\nu}_z})^2.
\end{aligned}$$

Two undetected neutrinos give rise to ambiguities (at most four fold)

Top mass assumed to be known $\Rightarrow$ 6 equations with 6 unknowns: p_{ν_x}, p_{ν_y}, p_{ν_z}, $p_{\bar{\nu}_x}$, $p_{\bar{\nu}_y}$, $p_{\bar{\nu}_z}$

What's the deal?

- Most precise solution is of major importance for measurements of top quark properties
  - top quark mass
  - $t\bar{t}$ spin correlations
- Interesting insights into solution characteristics/singularities
- Numerical methods can be gaged/compared
- Algebraic methods rediscovered for HEP
- Applicable for hadron collider (Tevatron, LHC) and lepton collider (NLC) environments

Transformation into polynomial equations

Initial system of non-linear equations can be transformed into two algebraic equations (=polynomials) with two unknowns p_{ν_x} and p_{ν_y} by means of ordinary algebraic operations.

$$
\begin{aligned}
f &= f_1 p_{\nu_y}^4 + f_2 p_{\nu_y}^3 + f_3 p_{\nu_y}^2 + f_4 p_{\nu_y} + f_5 \\
g &= g_1 p_{\nu_y}^4 + g_2 p_{\nu_y}^3 + g_3 p_{\nu_y}^2 + g_4 p_{\nu_y} + g_5
\end{aligned}
$$

where f and g are polynomials of the remaining unknowns p_{ν_x}, p_{ν_y} and the coefficients f_m, g_n are univariate polynomials of p_{ν_x}.

Technique of Resultants

Systems of equations consisting of two algebraic equations with two unknowns can be solved by means of Resultants:
The resultant can then be obtained by computing the determinant of the Sylvester matrix

$$\mathrm{Res}(p_{\nu_y}) \;=\; \mathrm{Det}\begin{pmatrix} f_1 & & & & g_1 & & & \\ f_2 & f_1 & & & g_2 & g_1 & & \\ f_3 & f_2 & f_1 & & g_3 & g_2 & g_1 & \\ f_4 & f_3 & f_2 & f_1 & g_4 & g_3 & g_2 & g_1 \\ f_5 & f_4 & f_3 & f_2 & g_5 & g_4 & g_3 & g_2 \\ & f_5 & f_4 & f_3 & & g_5 & g_4 & g_3 \\ & & f_5 & f_4 & & & g_5 & g_4 \\ & & & f_5 & & & & g_5 \end{pmatrix} = 0$$

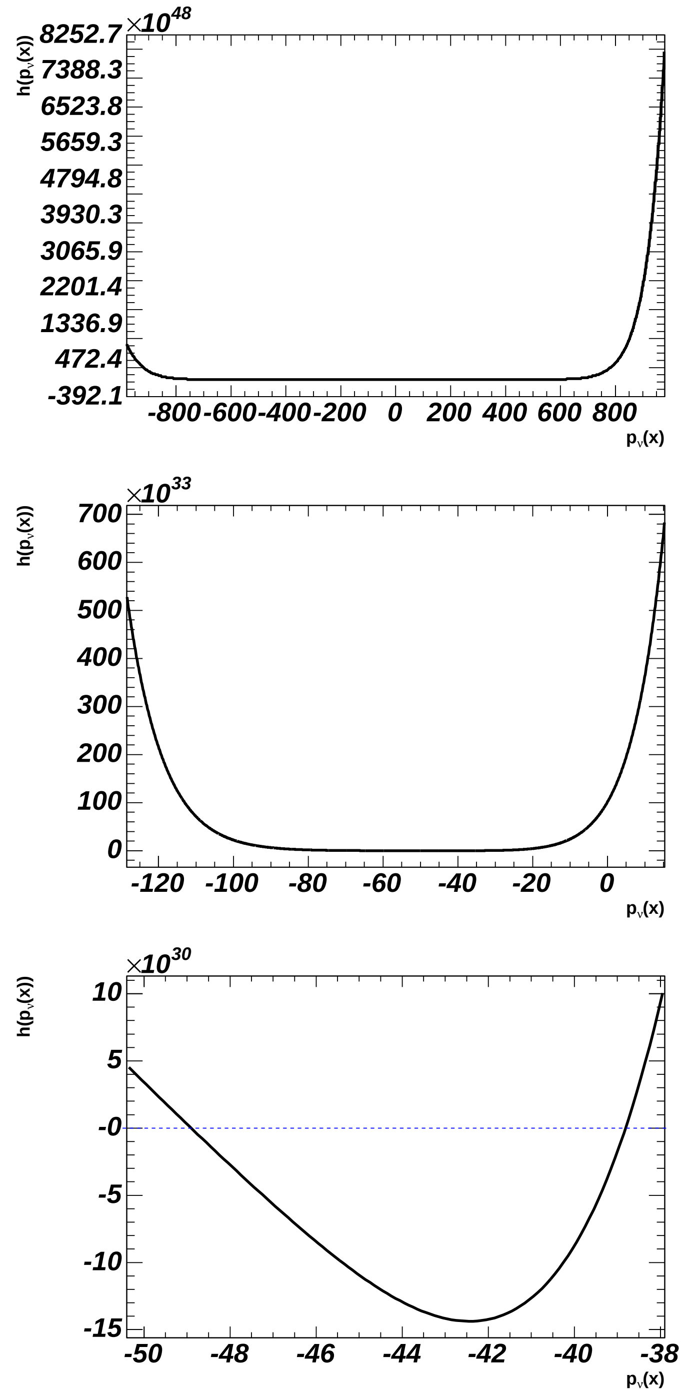
Technique of Resultants

The resultant is a univariate polynomial of degree 16
nd has the form:

$$
\begin{aligned}
0 \quad = \quad & h_1 p_{\nu_x}^{16} + h_2 p_{\nu_x}^{15} + h_3 p_{\nu_x}^{14} + h_4 p_{\nu_x}^{13} + h_5 p_{\nu_x}^{12} + h_6 p_{\nu_x}^{11} \\
+ \quad & h_7 p_{\nu_x}^{10} + h_8 p_{\nu_x}^{9} + h_9 p_{\nu_x}^{8} + h_{10} p_{\nu_x}^{7} + h_{11} p_{\nu_x}^{6} \\
+ \quad & h_{12} p_{\nu_x}^{5} + h_{13} p_{\nu_x}^{4} + h_{14} p_{\nu_x}^{3} + h_{15} p_{\nu_x}^{2} + h_{16} p_{\nu_x} + h_{17}
\end{aligned}
$$

vith the remaining unknown p_{ν_x}.

Free of any singularity (holds for all algebraic inter-
nediate steps involved)



Sturm's Theorem

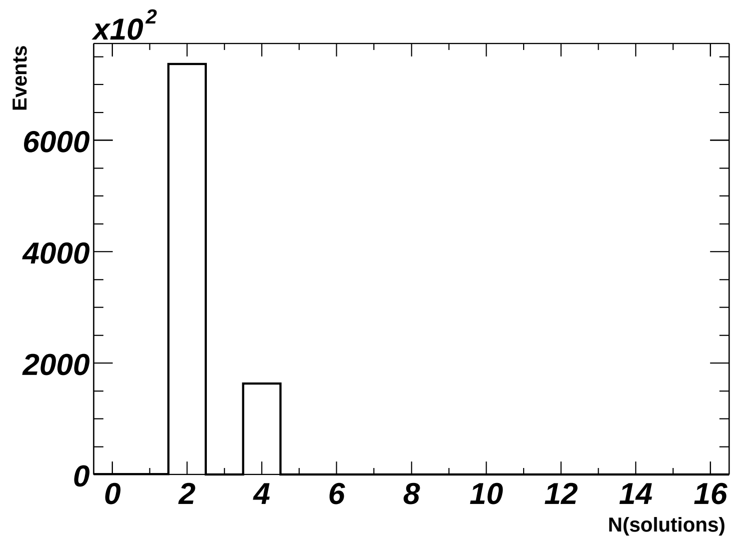
- In principle the problem can be reduced to an Eigenvalue problem.
  (Unfortunately implementations of Eigenvalue packages have problems with deg(h)≥ 14
- But even better: Number of solutions can be obtained analytically by applying Sturm's theorem which consists of building a sequence of univariate polynomials

$$\boxed{h(p_{\nu_x}), h'(p_{\nu_x}), h_a(p_{\nu_x}), h_b(p_{\nu_x}), ..., h_m(p_{\nu_x}) = \text{const.}} \; ,$$

- h' is the first derivative of the univariate polynomial h with respect to p_{ν_x}
- The following polynomials are the remainders of a long division of their immediate left neighbor polynomial divided by the next left neighbor polynomial.
- The sequence ends when the last polynomial is a constant. In the case the constant vanishes, the initial polynomial has at least one multiple real root which can be splitted by long division through the last non constant polynomial in the Sturm sequence. In this case one solution is already known.
- The sequence is evaluated at two neutrino momenta $p_{\nu_{x_{1,2}}}$
  (initially at the kinematic limits)
- Difference between the number of sign changes of evaluated sequence at the two interval limits is determined.
- Obtained quantity corresponds to the number of real solutions in the given interval.

Algorithm to obtain Solutions

- To reduce numerical inaccuracies, all polynomial evaluations are applied using Horner's rule which factors out powers of the polynomial variable p_{ν_x}.
- Further the solutions are separated by applying Sturm's theorem with varying interval boundaries.
- Once the solutions are separated in unique pairwise disjoint intervals they are polished by binary bracketing exploiting the knowledge about the sign change at the root in the given interval.



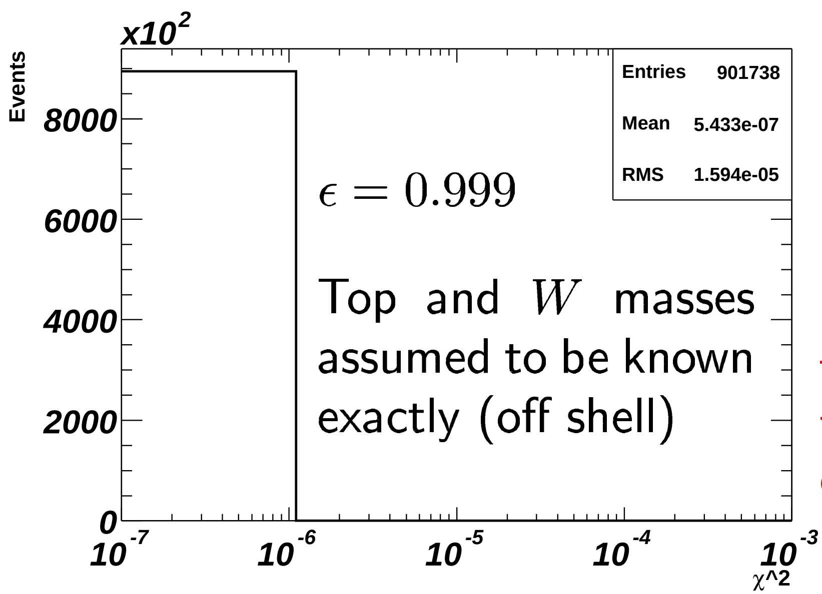
Algebraic Approach published in Physical Review D **72**, 095020 (2005)

Performance

n 99.9% of the events a solution can be found. The neutrino nomenta p_ν^{sol} of the solutions are compared to the generated nes p_ν^{gen} by defining a metric χ through

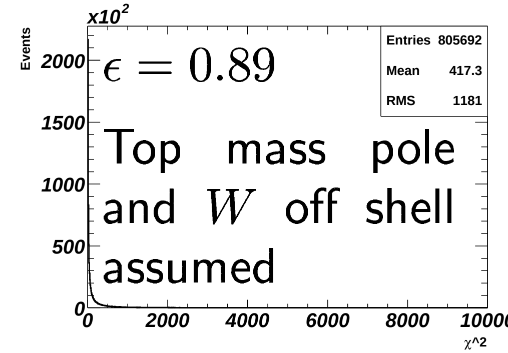
$$\begin{aligned}\chi^2 &= (p_{\nu_x}^{gen} - p_{\nu_x}^{sol})^2 + (p_{\nu_y}^{gen} - p_{\nu_y}^{sol})^2 + (p_{\nu_z}^{gen} - p_{\nu_z}^{sol})^2 \\ &+ (p_{\bar{\nu}_x}^{gen} - p_{\bar{\nu}_x}^{sol})^2 + (p_{\bar{\nu}_y}^{gen} - p_{\bar{\nu}_y}^{sol})^2 + (p_{\bar{\nu}_z}^{gen} - p_{\bar{\nu}_z}^{sol})^2 \ .\end{aligned}$$

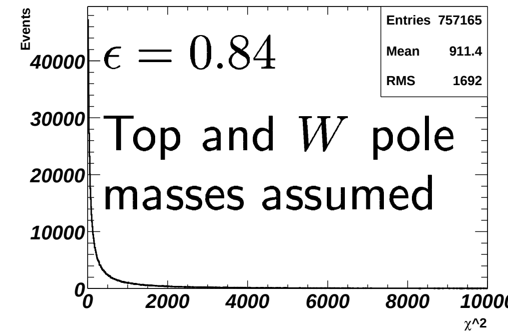
The solutions coincide in 99.7% of cases within real precision o the generated neutrino momenta.

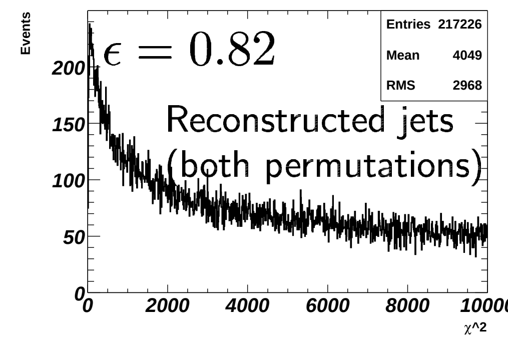


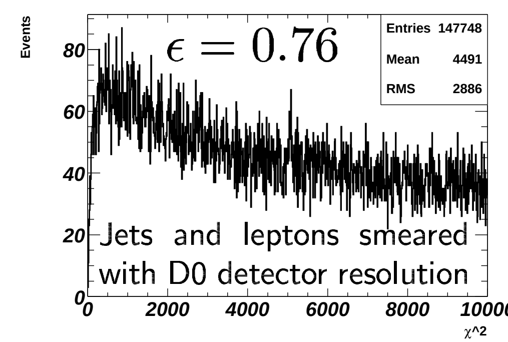
Minimal solution χ^2 per event distributions

General Numerical methods with higher efficiencies have to be cross checked!









Analytical Shortcut

- W boson mass constraint:

$$\begin{aligned} m_{W^+}^2 &= (E_{\ell+} + E_\nu)^2 - (\vec{p_{\ell+}} + \vec{p_\nu})^2 \\ &= E_{\ell+}^2 + 2E_{\ell+}E_\nu + E_\nu^2 - \vec{p_{\ell+}}^{\,2} - 2\vec{p_{\ell+}}\vec{p_\nu} - \vec{p_\nu}^{\,2} \\ &= m_{\ell+}^2 + 2E_{\ell+}E_\nu - 2\vec{p_{\ell+}}\vec{p_\nu} \end{aligned}$$

  can be rewritten as

$$E_\nu = \frac{m_{W^+}^2 - m_{\ell+}^2 + 2\vec{p_{\ell+}}\vec{p_\nu}}{2E_{\ell+}}.$$

- Top quark mass constraint:

$$E_\nu = \frac{m_t^2 - m_b^2 - m_{\ell+}^2 - 2E_bE_{\ell+} + 2\vec{p_b}\vec{p_{\ell+}} + 2(\vec{p_b} + \vec{p_{\ell+}})\vec{p_\nu}}{2(E_b + E_{\ell+})}$$

- Subtracting the two equations yields:

$$0 = a_1 + a_2 p_{\nu_x} + a_3 p_{\nu_y} + a_4 p_{\nu_z}$$

  (minimal Ansatz, others do exist and are pursued by others)

Analytical Shortcut

- Equating $E_\nu^2 = p_{\nu_x}^2 + p_{\nu_y}^2 + p_{\nu_z}^2$ and W boson mass constraint cancels accidentally $p_{\nu_z}^2$ dependencies.
- $\Rightarrow p_{\nu_z}$ can be eliminated in the linear equation.
  Similar for the anti-particle branch.
- Two polynomials of the two unknowns p_{ν_x} and p_{ν_y} of merely multidegree two can be formulated:

$$
\begin{aligned}
c &= c_0 p_{\nu_y}^2 + c_1 p_{\nu_y} + c_2, \\
d &= d_0 p_{\nu_y}^2 + d_1 p_{\nu_y} + d_2
\end{aligned}
$$

- The resultant

$$
\mathrm{Res}(p_{\nu_y}) = \mathrm{Det}\begin{pmatrix} c_0 & & d_0 & \\ c_1 & c_0 & d_1 & d_0 \\ c_2 & c_1 & d_2 & d_1 \\ & c_2 & & d_2 \end{pmatrix} = 0
$$

  yields a quartic

$$
h_0 p_{\nu_x}^4 + h_1 p_{\nu_x}^3 + h_2 p_{\nu_x}^2 + h_3 p_{\nu_x} + h_4
$$

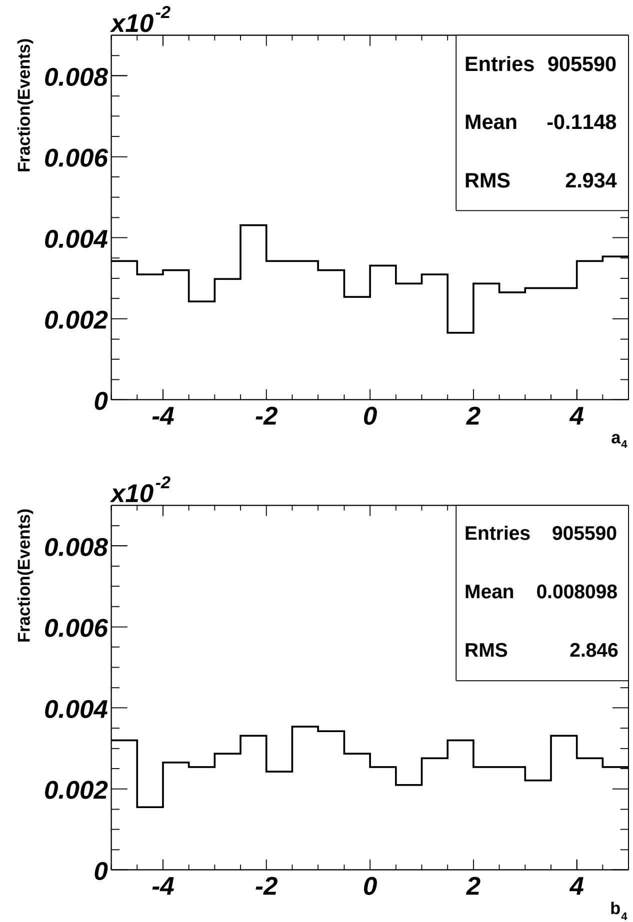
  which can be solved analytically.

Analytical Shortcut

- Coefficients c_m, d_n contain two reducible singularities which can be eliminated by multiplication of their Least Common Multiple.

- Two irreducible singularities are hidden in the linear equations when the coefficient in front of p_{ν_z} ($p_{\bar{\nu}_z}$) disappears so that the momentum cannot be determined.

- Ansatz of Algebraic Approach (free of any singularities) can be used to determine p_{ν_z} ($p_{\bar{\nu}_z}$).



Analytical Shortcut

- Fast
- No need for black box computer algebra
- Real Precision
- Confirms Performance of Algebraic Approach
- High solution efficiency can be re-established in repeated solution of same event resolution smeared. ($\Rightarrow$ Loss of Significance)

	$\frac{N_{\text{sol}=2}}{N_{\text{sol}>0}}$	$< N_{\text{sol}}^{>0} >$	RMS($N_{\text{sol}}^{>0}$)
t, W masses known exactly	0.82	2.37	0.77
W mass known exactly t pole mass assumed	0.84	2.32	0.74
t, W pole mass assumed	0.85	2.31	0.72
t, W pole mass assumed both $b\bar{b}$ permutations	0.59	3.00	1.35
reconstructed b-jets (parton matched)	0.79	2.42	0.82
wrong b-jet permutation (parton matched)	0.82	2.36	0.77
both b-jet permutations (parton matched)	0.52	3.22	1.48
both b-jet permutations (parton matched, jets + leptons smeared)	0.54	3.19	1.47
both b-jet permutations (parton matched, jets + leptons smeared), reconstructed objects 100 $\times$ resolution smeared	0.0072	7.96	4.72

Conclusions

- Interesting algebraic techniques introduced into HEP
- Algebraic Approach solves to real precision without singularity
- Minimal Analytical solution presented
  - faster than *Algebraic Approach*
  - confirms performance of *Algebraic Approach*
  - has singularities which can be removed
    in exploiting Ansatz of *Algebraic Approach*
- General numerical methods can be checked/gaged